# OKAWA'S THEOREM AND WIDE SUBCATEGORIES OF COHERENT SHEAVES ON CURVES

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In this note we give an alternative proof of a well-known result by Okawa [9]: the derived category of a smooth projective curve $X$ of genus $\geq 1$ has no non-trivial semi-orthogonal decompositions. Arguments in [9] are based on Serre duality and the fact that the canonical line bundle on $X$ is generated by global sections. Our proof is not using these facts, instead it grows from our study [6] and [5] of thick subcategories in $\operatorname{coh} X$.

For the sake of the reader, we make the arguments concise and self-contained, and avoid terminology from [6] and [5] that is not needed for our application.

Recall that an abelian category $\mathcal{A}$ is *hereditary* if $\operatorname{Ext}^i(A, B) = 0$ for any $A, B \in \mathcal{A}$ and $i \geq 2$. For example, the category $\operatorname{coh} X$ of coherent sheaves on a smooth curve is hereditary. The derived category $D^b(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$ has simple structure: any complex over $\mathcal{A}$ is isomorphic in $D^b(\mathcal{A})$ to the direct sum of its (shifted) cohomology objects. This allows one to relate thick (that is, idempotent-closed triangulated full) subcategories in $D^b(\mathcal{A})$ with certain subcategories in $\mathcal{A}$.

**Definition 1.** A full subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is called *wide* if $\mathcal{S}$ is closed under taking kernel, cokernels, and extensions.

Note that a wide subcategory is abelian, and the inclusion functor is exact.

**Proposition 2.** *Let $\mathcal{A}$ be an abelian hereditary category. Then there is a bijection between wide subcategories in $\mathcal{A}$ and thick subcategories in $D^b(\mathcal{A})$, given by assignments*

$$\mathcal{A} \supset \mathcal{S} \mapsto \langle \mathcal{S} \rangle \subset D^b(\mathcal{A}),$$
$$\mathcal{A} \supset \mathcal{T} \cap \mathcal{A} \mapsfrom \mathcal{T} \subset D^b(\mathcal{A}),$$

*where $\langle \mathcal{S} \rangle = \{F \in D^b(\mathcal{A}) \mid H^i(F) \in \mathcal{S} \quad \text{for all } i\}$ is the smallest triangulated subcategory in $D^b(\mathcal{A})$, containing $\mathcal{S}$.*

*Proof.* This is [3, Th. 5.1] or [7, Prop. 4.4.17]. □

Recall that an abelian category is said to be *finite length* if any object of $\mathcal{A}$ has a finite filtration with simple factors. By Jordan — Hölder Theorem, the collection of such simple factor doesn't depend on the choice of filtration. Therefore, the Grothendieck group $K_0(\mathcal{A})$ is freely generated by isoclasses of simple objects in $\mathcal{A}$.

We present a new proof of the following theorem by Okawa [9].

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**Theorem 3.** *Let $X$ be a smooth connected projective curve over a field, and*

$$D^b(\operatorname{coh} X) = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$$

*be a semi-orthogonal decomposition. Then $\mathcal{T}_1 = 0$ or $\mathcal{T}_2 = 0$.*

*Proof.* Recall that any sheaf on $X$ is a direct sum of a torsion sheaf and a torsion-free sheaf. In this proof we call a triangulated subcategory $\mathcal{T} \subset D^b(\operatorname{coh} X)$ *torsion* (resp. *torsion-free*) if all sheaves in $\mathcal{T} \cap \operatorname{coh} X$ are torsion (resp. torsion-free).

Assume $\mathcal{T}_1, \mathcal{T}_2 \neq 0$, let $\mathcal{S}_i = \mathcal{T}_i \cap \operatorname{coh} X$ be the corresponding wide subcategories in $\operatorname{coh} X$. By Proposition 2, $\mathcal{S}_1, \mathcal{S}_2 \neq 0$. We claim that both $\mathcal{T}_1, \mathcal{T}_2$ are torsion-free. Indeed, assume there is a non-zero torsion sheaf $F \in \mathcal{T}_1$. Then for any non-zero torsion-free sheaf $V$ one has $\operatorname{Hom}(V, F) \neq 0$. Hence $V \notin \mathcal{T}_2$ and $\mathcal{T}_2$ is torsion. Take a non-zero torsion sheaf $F_2 \in \mathcal{T}_2$, then $\operatorname{Ext}^1(F_2, V) \cong \operatorname{Hom}(V, F_2 \otimes \omega_X)^* \cong \operatorname{Hom}(V, F_2)^* \neq 0$ by Serre duality. Hence $V \notin \mathcal{T}_1$ and $\mathcal{T}_1$ is torsion. We deduce that $D^b(\operatorname{coh} X)$ is generated by torsion sheaves (use Proposition 2) and get a contradiction.

Now we know that $\mathcal{T}_1, \mathcal{T}_2$ are torsion-free. We claim that abelian categories $\mathcal{S}_i$ are finite length categories. Take an object $F \in \mathcal{S}_i$ and prove that $F$ has a finite filtration with simple (in $\mathcal{S}_i$) factors by induction in the rank $r(F)$ of the vector bundle $F$. Indeed, if $r([F]) = 0$ then $F = 0$ and there is nothing to prove. Assuming $F$ is not simple in $\mathcal{S}_i$, there is a non-trivial exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathcal{S}_i$. Here $r([F]) = r([F']) + r([F''])$, $r([F']), r([F'']) > 0$, and $r([F']), r([F'']) < r([F])$. By induction $F', F''$ have a filtration with simple factors, hence $F$ also does.

Next, we note that the number of isoclasses of simple objects in $\mathcal{S}_1, \mathcal{S}_2$ is finite. Indeed, $D^b(\operatorname{coh} X)$ has a classical generator (see [2]), thus admissible subcategories $\mathcal{T}_i$ also do. Replacing a complex by the direct sum of its cohomology sheaves, one can assume that there are classical generators $G_i \in \mathcal{S}_i$ for $\mathcal{T}_i$. It is easy to see now that any object in $\mathcal{S}_i$ has a filtration with simple factors being simple factors of $G_i$, and there are finitely many of these. We conclude that the Grothendieck group $K_0(\mathcal{S}_i) \cong K_0(\mathcal{T}_i)$ is finitely generated for $i = 1, 2$, and $K_0(D^b(\operatorname{coh} X)) \cong K_0(\mathcal{T}_1) \oplus K_0(\mathcal{T}_2)$ is also finitely generated. But this is impossible for a curve of genus $\geq 1$. □

In fact, Theorem 3 is a special case of a more general statement for the derived category of a smooth orbifold curve, obtained in [5]. Consider a smooth orbifold $\mathbb{X}$, whose underlying scheme is a smooth projective curve $X$. Such $\mathbb{X}$ can be constructed from $X$ be applying so called *root construction* at some points $x_1, \ldots, x_n \in X$ with multiplicities $r_1, \ldots, r_n \geq 2$. See [4] or [1, Sect. 3] for stack-theoretic definition of $\mathbb{X}$ and [8, Sect. 4] for elementary representation-theoretic description of the resulting category of $\operatorname{coh} \mathbb{X}$ coherent sheaves on $\mathbb{X}$. In contrast to the smooth case, $\operatorname{coh} \mathbb{X}$ contains exceptional torsion sheaves, supported at orbifold points $x_1, \ldots, x_n$. The following theorem from [5] essentially says that all semi-orthogonal decompositions of $D^b(\operatorname{coh} \mathbb{X})$ are due to such exceptional torsion sheaves.

**Theorem 4.** *Let $\mathbb{X}$ be a smooth connected projective orbifold curve whose underlying curve has genus $\geq 1$. Assume $D^b(\operatorname{coh} \mathbb{X}) = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ is a semi-orthogonal decomposition. Then* *either* $\mathcal{T}_1$ *or* $\mathcal{T}_2$ *is generated by an exceptional collection of torsion sheaves. Moreover, there are finitely many subcategories in* $D^b(\operatorname{coh}\mathbb{X})$ *generated by an exceptional collection of torsion sheaves, and they can be explicitly classified.*

## References

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